A Review of CAS Mathematical Capabilities

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Abstract

Computer algebra systems (CASs) have become an important computational tool in the last decade. General purpose CASs, which are designed to solve a wide variety of problems, have gained special prominence. In this paper, the capabilities of six major general purpose CASs (Axiom, Derive, Macsyma, Maple, Mathematica and Reduce) are reviewed on 123 short problems covering a broad range of (primarily) symbolic mathematics.

A demo was developed for each CAS, run and the results evaluated. Problems were graded in terms of whether it was easy or difficult or possible to produce an answer and if an answer was produced, whether it was correct. It is the author's hope that this review will encourage the development of a comprehensive CAS test suite.

Presented below is a summary of 123 mathematical problems (primarily symbolic) that were given to the six general purpose computer algebra systems (CASs) listed in Table 1. The CAS versions tested were those that were available to the author and were typically the newest and most comprehensive versions that were generally available at the time of this evaluation. The one exception was Derive as the Professional Version has some capabilities not present in the Student Edition that would have probably resulted in additional successes.

The notations used in the summary are explained in Table 2. A demo was developed for each CAS, run and the results evaluated. Problems easily and successfully solved are marked by a $\bullet$. Those that required more effort than a mere 'simplify' (sometimes considerably more effort) or the results could not be completely simplified or were incomplete in some way are marked with an $\circ$. If the problem could not be solved by the CAS, the corresponding entry in the summary table was left blank. Incorrect answers are indicated by an $\times$.

Problem descriptions are abbreviated due to space limitations with this format and may not always be complete. However, the complete demos and their output are readily obtained by anonymous FTP from `math.unm.edu`. The directory `pub/cas` will contain current versions of these and other potentially interesting CAS related files.

The philosophy that I followed in making these comparisons included several facets. The choice of problems was deliberately broad as a primary goal was to provide a useful indication of the breadth of coverage of each general purpose system. A secondary goal was to give a feeling of the depth of coverage provided for certain classes of problems. This aspect of the survey is certainly incomplete as right now some areas of mathematics are covered by a more interesting range of examples than others (e.g., the problems for matrix algebra, products and limits currently are all quite easy). It is my intention to remedy this unevenness in a future version of this review.

The primary emphasis of this review is on exact, symbolic mathematics, although a few approximate, numerical problems are also included. I have totally ignored for the time being issues of graphics, language design, user interface and computational speed, all of which are important as well. The six general purpose CASs examined can, with varying amounts of user assistance, solve a great variety of problems. I have tried to emphasize those problems that ideally should involve minimal user intervention.

If it was necessary to tell a system most of the steps to solve a problem, I did not consider this a solution by the CAS.[1] Poorly documented methods and results were also judged unfavorably. If only 'some' user intervention was required, this was acknowledged by an $\circ$. If at most minimal aid was provided by the user, only then was a $\bullet$ rewarded. Packages not provided with the standard distribution of a CAS were excluded from consideration. Many systems do, however, have some quite nice user packages that are only available (easily) on the Internet/via email or must be purchased separately.

Some final comments: in order to perform this review, it was necessary to dig deeply into each of the CASs' reference manuals. None of them were particularly easy to use without a great deal of study and unfortunately, the indices in general were of fairly limited utility (Mathematica's was the best of the lot). Other people have also reviewed the software in or made comparisons of multiple CASs, emphasizing different aspects than was done here. For example, see [Har91, Her94, Sim92]. In summary, the general purpose CASs surveyed here are all quite powerful, each with its own particular strengths and weaknesses. This review should only be used as a guide, not as a definitive basis for comparison. Comments are welcome and additional problems are actively solicited.

References

[Har91] David Harper, Chris Wooff and David Hodgkinson, *A Guide to COMPUTER ALGEBRA SYSTEMS*, John Wiley & Sons, 1991.

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[Rob93] Nicolas Robidoux, "Does Axiom Solve Systems of O.D.E.'s Like Mathematica?", LA-UR-93-2235, Los Alamos National Laboratory, Los Alamos, New Mexico.

[Sim92] Barry Simon, "Comparative CAS Reviews", *Notices of the American Mathematical Society*, Volume 39, Number 7, September 1992, 700–710.

[1] However, after receiving some artful responses from various CAS developers, I may institute a 'super tricky' category in a future version of this review!

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Axi	Axiom 1.2	March 17, 1993
Der	Derive 2.0 Student Edition	1992
Mac	Macsyma 418.11	1993
Mpl	Maple V Release 3	1994
Mma	Mathematica 2.2	1993
Red	Reduce 3.5	15–Oct–93

Table 1 CASs used.

$\bullet$	success! (hurrah)
$\circ$	success, but indirectly, incomplete or unsimplified
	could not do the problem (boo)
$\times$	produced the wrong answer (hiss)
$\Rightarrow$	yields
$\rightarrow$	then
”	previous result
D	derivative operator
diff	differentiate
factor($\ldots$, α)	factor over the algebraic field extension α
$J_\mu(z)$	Bessel function of the first kind of order μ
${K_i}^h{}_{jk\vert\ell}$	covariant derivative of the Riemann curvature tensor
laplace($f(t)$, $t \rightarrow s$)	Laplace transform of $f(t)$ under $t \rightarrow s$
N($\ldots\{, k\}$)	numerically evaluate (to k-digit precision)
(operator)	define as an operator
pade($f(x)$, $x = a$)	Pade approximation of $f(x)$ about $x = a$
poly	polynomial
power_series($f(x)$, $x = a$)	general power series formula of $f(x)$ about $x = a$
(real)	produce an explicitly real solution
rectform($f(z)$)	rectangular form of the complex function $f(z)$
(rewrite rules)	using user supplied rewrite rules
$\sigma(\ldots)$	sample standard deviation
solve($x = f(y)$, $y = a$, series)	solve for $y(x)$ about $y = a$ using series reversion
taylor($f(x)$, $x = a$)	truncated Taylor series of $f(x)$ about $x = a$
$\zeta(n)$	Riemann zeta function

Table 2 Notations used.

#	PROBLEM	Axi	Der	Mac	Mpl	Mma	Red
1	50!	●	●	●	●	●	●
2	factor(50!)	●	●	●	●	●	○
3	$\frac{1}{2} + \cdots + \frac{1}{10} \Rightarrow \frac{4861}{2520}$	●	●	●	●	●	●
4	$\mathsf{N}(e^{\pi\sqrt{163}}, 50) \approx 262537412640768743.0$	●	$^{\mathbf{1}}$	●	●	●	●
5	$\mathsf{N}(J_2(1+i)) \approx 0.04158 + 0.24740i$	●		●		●	●
6	$\mathsf{N}(\frac{1}{7}) \Rightarrow 0.\overline{142857}$	●					
7	continued fraction of 3.1415926535	●		●	●	●	
8	$\sqrt{2\sqrt{3}+4} \Rightarrow 1 + \sqrt{3}$		●	●	●		●
9	$\sqrt{14 + 3\sqrt{3 + \cdots}} \Rightarrow 3 + \sqrt{2}$		●	○	●		
10	$2\infty - 3 \Rightarrow \infty$	●		○	○	●	
11	$\sigma([1,2,3,4,5]) \Rightarrow \sqrt{\frac{5}{2}}$		$\circ^{\mathbf{2}}$	●	●	●	
12	hypothesis testing: t distribution			○	○	●	
13	hypothesis testing: normal distribution		$\circ^{\mathbf{2}}$	○	○	●	
14	$\frac{x^2-4}{x^2+4x+4} \Rightarrow \frac{x-2}{x+2}$	●	●	●	●	●	●
15	$\frac{e^x-1}{e^{x/2}+1} \Rightarrow e^{x/2} - 1$	●	●	●	○	●	
16	$\mathsf{expand}((x+1)^{20}) \rightarrow \mathsf{diff} \rightarrow \mathsf{factor}$	●	●	●	●	●	●
17	$\mathsf{factor}(x^{100} - 1)$	●	●	●	●	●	●
18	$\mathsf{factor}(x^4 - 3x^2 + 1,\ \mathsf{RootOf}(\phi^2 - \phi - 1))$	●		●	●		
19	$\mathsf{factor}(x^4 - 3x^2 + 1)$ mod 5	●		●	●	●	●
20	$\frac{x^2+2x+3}{x^3+4x^2+5x+2} \Rightarrow \frac{3}{x+2} - \frac{2}{x+1} + \frac{2}{(x+1)^2}$	○	●	●	●	●	●
21	$\mathsf{assume}(x \geq y,\ y \geq z,\ z \geq x);\quad \mathsf{is}(x = z)$			●	×		
22	$\mathsf{assume}(x > y,\ y > 0);\quad \mathsf{is}(2x^2 > 2y^2)$			●	●		
23	$\frac{\cos 3x}{\cos x} \Rightarrow \cos^2 x - 3\sin^2 x$ (or similar)	○	●	●	●	●	
24	$\frac{\cos 3x}{\cos x} \Rightarrow 2\cos 2x - 1$	○	●	●	●	●	
25	$\frac{\cos 3x}{\cos x} \Rightarrow \cos^2 x - 3\sin^2 x$ (rewrite rules)	●		●	$\circ^{\mathbf{3}}$	●	●
26	$\sqrt{997} - (997^3)^{1/6} \Rightarrow 0$	●	●	●	●	●	○
27	$\sqrt{999983} - (999983^3)^{1/6} \Rightarrow 0$	●	●	●	●		○
28	$(2^{1/3} + 4^{1/3})^3 - 6(2^{1/3} + 4^{1/3}) - 6 \Rightarrow 0$		●	●	●	●	●
29	$x^{1/n}y^{1/n} - (xy)^{1/n} \Rightarrow 0 \quad (x, y > 0)$	○	●	●	●	○	○
30	$\log(\tan(\frac{1}{2}x + \frac{\pi}{4})) - \sinh^{-1}(\tan x) \Rightarrow 0$			○	○		
31	derivative of above is 0 & above at 0 is 0	○	○	●	●	○	
32	$\log \frac{2\sqrt{r}+1}{\sqrt{4r+4\sqrt{r}+1}} \Rightarrow 0$		●	●		●	●
33	$(4r + 4\sqrt{r} + 1)^{\frac{\sqrt{r}}{2\sqrt{r}+1}}(2\sqrt{r}+1)^{\frac{1}{2\sqrt{r}+1}} \cdots$	●	●			○	

#	PROBLEM	**Axi**	**Der**	**Mac**	**Mpl**	**Mma**	**Red**
34	$\mathsf{rectform}(\log(3+4i)) \Rightarrow \log 5 + i\tan^{-1}\frac{4}{3}$	○	○	●	●	○	
35	$\mathsf{rectform}(\tan(x+iy))$	○	○	●	●	●	
36	$\frac{\sqrt{xy\|z\|^2}}{\sqrt{x}\|z\|} \Rightarrow \frac{\sqrt{xy}}{\sqrt{x}} \not\Rightarrow \sqrt{y}$		●	○	●		×
37	$\sqrt{\frac{1}{z}} - \frac{1}{\sqrt{z}} \not\Rightarrow 0$	×	●	●	●	●	×
38	$\log e^z \not\Rightarrow z$	●	×	×	●	●	×
39	$\log e^{10i} \Rightarrow (10-4\pi)i$ (principal value)		●		○	○	
40	$\tan^{-1}(\tan z) \not\Rightarrow z$	×	●	○	●	●	●
41	$\sqrt{e^z} - e^{z/2} \not\Rightarrow 0$	●	×	●	●	●	×
42	$\frac{x=0}{2} + 1 \Rightarrow \frac{x}{2} + 1 = 1$		●	●		×	×
43	$\mathsf{solve}(3x^3 - 18x^2 + 33x - 19 = 0)$ (real)			●	●	●	●[4]
44	$\mathsf{solve}(x^4 + x^3 + x^2 + x + 1 = 0)$	○		●	●	●	●
45	verify a solution of the above	●		●	●	●	●
46	$\mathsf{solve}(e^{2x} + 2e^x + 1 = z,\, x)$	●	●	●	●	●	●[5]
47	$\mathsf{solve}((x+1)(\sin^2 x + 1)^2 \cos^3 3x = 0)$			○	○	○	●[5]
48	$\mathsf{solve}(e^z = 1) \Rightarrow z = 0\,[\,+n2\pi i]$	●	●	●	●	●	●[5]
49	$\mathsf{solve}(\sin x = \cos x) \Rightarrow x = \frac{\pi}{4}\,[\,+n\pi]$	●	●		●		○
50	$\mathsf{solve}(\tan x = 1) \Rightarrow x = \frac{\pi}{4}\,[\,+n\pi]$	●	●	●	●	●	●[5]
51	$\mathsf{solve}(\sin x = \tan x) \Rightarrow 0, 0\,[\,+n\pi,\,+n2\pi]$	○	○	●	●		○
52	$\mathsf{solve}(\sqrt{x^2+1} = x - 2) \Rightarrow x = \{\}$	×	●		○	●	×
53	solve a 3×3 dependent linear system	●	●	●	●	●	●
54	solve a system of nonlinear equations	●		●	●	●	●
55	invert a 2×2 symbolic matrix	●	●	●	●	●	●
56	$\mathsf{det}(4 \times 4$ Vandermonde matrix$)$	●	●	●	●	●	●
57	eigenvalues of a 3×3 integer matrix	●	●	●	●	●	●
58	tensor covariant derivative			●			
59	$K_i{}^h{}_{jk\|\ell} + K_i{}^h{}_{k\ell\|j} + K_i{}^h{}_{\ell j\|k} \Rightarrow 0$ (Bianchi)			●			
60	$\sum_{k=1}^n k^3 \Rightarrow \frac{n^2(n+1)^2}{4}$	●	●	●	●	●	●
61	$\sum_{k=1}^\infty (\frac{1}{k^2} + \frac{1}{k^3}) \Rightarrow \frac{\pi^2}{6} + \zeta(3)$			●	●	●	
62	$\mathsf{N}(\sum_{k=1}^\infty (\frac{1}{k^2} + \frac{1}{k^3})) \approx 2.84699$			●	●	●	
63	$\prod_{k=1}^n k \Rightarrow n!$		●	●	●	●	
64	$\lim_{n\to\infty}(1+\frac{1}{n})^n \Rightarrow e$; $\lim_{x\to 0}\frac{1-\cos x}{x^2} \Rightarrow \frac{1}{2}$	●	●	●	●	●	●
65	$\frac{d^2}{dt^2} y(x(t)) \Rightarrow \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{dy}{dx}\frac{d^2x}{dt^2}$	●		●	●	●	
66	$\int \frac{1}{x^3+2}\,dx \rightarrow \mathsf{diff} \rightarrow \mathsf{simplify}$	●	●	●	●	●	●

#	PROBLEM	Axi	Der	Mac	Mpl	Mma	Red
67	$\int \frac{1}{a+b\cos x}\,dx \quad (a<b)$	$\bullet$		$\bullet$	$\bullet$		
68	$\frac{d}{dx}\int \frac{1}{a+b\cos x}\,dx = \frac{1}{a+b\cos x}$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\circ$
69	$\frac{d}{dx}\lvert x\rvert \Rightarrow \frac{x}{\lvert x\rvert}$ or $\mathrm{sign}(x)$	$\bullet$	$\bullet$	$\bullet$	$\circ$[6]		$\bullet$
70	$\int \lvert x\rvert\,dx \Rightarrow \frac{x\lvert x\rvert}{2}$		$\bullet$	$\bullet$	$\bullet$		$\bullet$
71	$\frac{d}{dx}\lvert x\rvert$ (piecewise defined)	$\times$		$\bullet$	$\bullet$	$\bullet$	
72	$\int \lvert x\rvert\,dx$ (piecewise defined)	$\times$		$\bullet$			
73	$\int \frac{x}{\sqrt{1+x}+\sqrt{1-x}}\,dx \Rightarrow \frac{(1+x)^{3/2}+(1-x)^{3/2}}{3}$	$\bullet$	$\bullet$	$\circ$	$\bullet$	$\bullet$	$\bullet$
74	$\int \frac{\sqrt{1+x}-\sqrt{1-x}}{2}\,dx \Rightarrow \frac{(1+x)^{3/2}+(1-x)^{3/2}}{3}$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
75	$\int_{-1}^{1} \frac{1}{x}\,dx \Rightarrow 0$ (principal value)		$\times$	$\bullet$	$\circ$	[7]	
76	$\int_{-1}^{1} \frac{1}{x^2}\,dx \Rightarrow$ divergent		$\times$	$\bullet$	$\bullet$		
77	$\int_0^1 \sqrt{x+\frac{1}{x}-2}\,dx \Rightarrow \frac{4}{3}$	[8]	$\bullet$	$\bullet$	$\bullet$	$\bullet$	
78	$\int_1^2 \sqrt{x+\frac{1}{x}-2}\,dx \Rightarrow \frac{4-\sqrt{8}}{3}$	$\circ$	$\bullet$	$\bullet$	$\times$	$\bullet$	
79	$\int_0^2 \sqrt{x+\frac{1}{x}-2}\,dx \Rightarrow \frac{8-\sqrt{8}}{3}$	[8]	$\bullet$		$\times$	$\bullet$	
80	$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx \quad (a>0) \Rightarrow \frac{\pi}{a}e^{-a}$			$\bullet$	$\times$[9]	$\circ$[9]	
81	$\int_0^{\infty} \frac{t^{a-1}}{1+t}\,dt \quad (0<a<1) \Rightarrow \frac{\pi}{\sin \pi a}$			$\bullet$	$\circ$[9]	$\bullet$[9]	
82	$\int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} 1\,dz\,dy\,dx \Rightarrow \frac{abc}{6}$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	
83	$\mathsf{taylor}(\frac{1}{\sqrt{1-(v/c)^2}},\ v=0)$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
84	$\frac{1}{\mathrm{above}^2} \Rightarrow 1-\frac{v^2}{c^2}+\cdots$	$\bullet$	$\circ$	$\bullet$	$\circ$	$\bullet$	$\circ$
85	$\frac{\mathsf{taylor}(\sin x,\, x=0)}{\mathsf{taylor}(\cos x,\, x=0)} = \mathsf{taylor}(\tan x,\ x=0)$	$\bullet$	$\circ$	$\bullet$	$\circ$	$\bullet$	$\circ$
86	$\mathsf{taylor}((\log x)^a e^{-bx},\ x=1)$			$\bullet$			$\circ$
87	$\mathsf{taylor}(\log(\sinh z)+\log(\cosh(z+w)))$			$\bullet$	$\circ$[10]	$\circ$[10]	$\bullet$
88	" $-$ $\mathsf{taylor}(\log(\sinh z \cosh(z+w)),\ z=0)$			$\bullet$	$\circ$[10]	$\circ$[10]	$\bullet$
89	$\mathsf{taylor}(\log(\frac{\sin x}{x}),\ x=0)$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
90	$\mathsf{power\_series}(\log(\frac{\sin x}{x}),\ x=0)$			$\bullet$			
91	$\mathsf{power\_series}(e^{-x}\sin x,\ x=0)$			$\circ$	$\bullet$		
92	$\mathsf{solve}(x=\sin y+\cos y,\ y=0,$ series$)$			$\bullet$	$\bullet$	$\bullet$	
93	$\mathsf{pade}(e^{-x},\ x=0) \Rightarrow \frac{2-x}{2+x}$	$\bullet$		$\circ$	$\bullet$	$\bullet$	
94	$\mathsf{laplace}(\cos((\omega-1)t),\ t\to s) \Rightarrow \frac{s}{s^2+(\omega-1)^2}$	$\bullet$	$\circ$[11]	$\bullet$	$\bullet$	$\bullet$	$\bullet$
95	inverse Laplace transform of above	$\bullet$		$\bullet$	$\bullet$	$\bullet$	$\circ$
96	$\mathsf{solve}([r_{n+2}-2r_{n+1}+r_n=2,\ \ldots],\ r_n)$			$\circ$	$\bullet$	$\bullet$	
97	$\mathsf{solve}([\frac{d^2f}{dt^2}+f=\sin t,\ f(0)=f'(0)=0])$	$\bullet$	$\circ$[12]	$\bullet$	$\circ$	$\bullet$	$\circ$
98	above solution using Laplace transforms			$\bullet$	$\bullet$	$\bullet$	
99	$\mathsf{solve}(x^2\frac{dy}{dx}+3xy=\frac{\sin x}{x},\ y(x))$	$\bullet$	$\circ$[12]	$\bullet$	$\bullet$	$\bullet$	$\bullet$

#	PROBLEM	Axi	Der	Mac	Mpl	Mma	Red
100	solve($\frac{d^2y}{dx^2} + y(\frac{dy}{dx})^3 = 0,\ y(x)$)			•	•		
101	solve($\frac{d}{dx}y(x,a) = ay(x,a),\ y(x,a)$)			•	[13]	[13]	•
102	solve($[\frac{d^2y}{dx^2} + k^2y = 0,\ y(0) = 0,\ y'(1) = 0]$)				[14]	[14]	
103	solve($[\frac{dx}{dt} = x - y,\ \frac{dy}{dt} = x + y],\ [x,\ y]$)			•	•	•	
104	verify the above is a solution			•	•	○	
105	solve($[\frac{dx}{dt} = x\left(1 + \frac{\cos t}{2+\sin t}\right),\ \frac{dy}{dt} = x - y]$)						
106	as above, but one equation at a time	○	○	○	•	○	•
107	$L = (D-1)(D+2)$ (operator)	•		•	•	•	
108	$L(f) \Rightarrow D^2f + Df - 2f$	•		•	•	○	
109	$L_y(g(y)) \Rightarrow \frac{d^2g}{dy^2} + \frac{dg}{dy} - 2g$	○		•	•	○	
110	$L_z(A\sin z^2)$	○		•	•	○	
111	$T = \sum_{k=0}^{2} \frac{(D^kf)(a)}{k!}(x-a)^k$ (operator)	○		•	•	•	○
112	$T(f) \Rightarrow f(a) + (Df)(a)(x-a) + \cdots$	•		•	•	•	○
113	$T_{y,b}(g(y)) \Rightarrow g(b) + \frac{dg}{dy}\|_{y=b}(y-b) + \cdots$	•		•	•	•	○
114	$T_{z,c}(\sin z)$	•		•	•	•	•
115	compute Legendre polys directly	•	•	•	○	•	•
116	compute Legendre polys recursively	•	•	•	•[15]	•	•
117	evaluate the 4^{th} Legendre poly at 1	○	○	•	•	○	○
118	$p = \sum_{i=1}^{5} a_i x^i$	•	•	•	•	•	•
119	Horner's rule applied to the above			•	•	[16]	
120	convert to FORTRAN syntax	○		•	•	○	○
121	true and false $\Rightarrow$ false	•		•	•	•	○
122	x or (not x) $\Rightarrow$ true				•	•	
123	x or y or (x and y) $\Rightarrow$ x or y				•		

[1] computes value to 5 digit accuracy
[2] population ($\sqrt{n}$) rather than a sample ($\sqrt{n-1}$) standard deviation used
[3] pattern matching is primitive
[4] this happens automatically!
[5] produces complete general solution!
[6] lack of documentation on the answer that is returned
[7] can be shown numerically with some help
[8] incorrect answer produced with noPole option
[9] ignores assumptions on a
[10] limiting the order to 0 or 1 (Maple) or 1 or 2 (Mathematica) produces errors
[11] answer contains an unevaluated (but obvious) limit
[12] requires some user sophistication to solve
[13] claims that this is a partial differential equation!
[14] produces only the trivial solution
[15] tricky!—order of statements can be critical
[16] definition is 'hidden' within one of the standard packages